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LETTER TO THE EDITOR

On solutions of curl $a = ka$ and force-free magnetic fields

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Abstract. In a recent letter, Salingaros has claimed that solutions of the equation curl $\boldsymbol{a} = k\boldsymbol{a}$, k = constant, for a force-free magnetic field are not invariant with respect to gauge transformations and lack covariance with respect to general transformations. He has suggested that the force-free condition may not be used in physical models. However, in this letter, we show that the solutions are invariant to gauge transformations and we argue that solutions of Maxwell's equations which satisfy the force-free condition locally in a restricted domain are applicable to real physical situations.

In some physical models, notably in astrophysics, the magnetic field $\boldsymbol{B}$ is required to satisfy, in some region of space, the force-free condition

$$\text{curl}\,\boldsymbol{B} = \alpha\boldsymbol{B} \tag{1}$$

where α is a scalar function. Solutions of this equation with $\alpha \equiv k$, a constant, are known and have been proposed as possible representations of magnetic fields in physical models (e.g. Ferraro and Plumpton 1966). Recently Salingaros (1986) has questioned the validity of using these solutions in physical models.

Consider the equation

$$\text{curl}\,\boldsymbol{a} = k\boldsymbol{a} \qquad k = \text{constant}. \tag{2}$$

Salingaros has argued that solutions of equation (2) are

(i) not gauge invariant—a requirement which must be satisfied by $\boldsymbol{B}$—and

(ii) not covariant with respect to general transformations, thus excluding them from physical applications.

We shall examine each of these claims.

Firstly, we shall show that solutions of $\boldsymbol{B}$ satisfying equation (2) do not violate the requirement of gauge invariance. There is an inconsistency in the equations used by Salingaros which has led him to conclude otherwise. If $\boldsymbol{A}$ is the magnetic vector potential such that

$$\boldsymbol{B} = \text{curl}\,\boldsymbol{A} \tag{3}$$

then from equations (2) and (3) we have

$$\boldsymbol{A} = \boldsymbol{B}/k + \nabla\varphi \tag{4}$$

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where φ is a scalar function satisfying div $\boldsymbol{A} = \nabla^2\varphi$. Now, the test of gauge invariance is whether the magnetic field intensity remains unchanged under a gauge transformation, i.e. under the addition of the gradient of an arbitrary scalar function to the vector potential $\boldsymbol{A}$ (Landau and Lifshitz 1975). Let

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\lambda = \boldsymbol{B}/k + \nabla(\varphi + \lambda) \tag{5}$$

where λ is an arbitrary scalar function. Then equations (2), (4) and (5) give

$$\boldsymbol{B}' = \text{curl}\ \boldsymbol{A}' = \text{curl}(\boldsymbol{B}/k) = \boldsymbol{B}. \tag{6}$$

Thus we see that the magnetic field vector $\boldsymbol{B}$ satisfying equation (2) is gauge invariant. The error in Salingaros' conclusion is due to an incorrect interpretation of equation (3*b*) in his letter. The expression for $\boldsymbol{B}'$ should contain φ' in place of φ giving $\nabla\varphi' = \nabla(\varphi + \lambda)$ or $\varphi' = \varphi + \lambda +$ constant, which is the transformation equation for φ.

In fact, gauge invariance becomes more obvious when equation (2) is written in terms of the vector potential. We have

$$\text{curl curl}\ \boldsymbol{A} = k\ \text{curl}\ \boldsymbol{A}.$$

Under the gauge transformation $\boldsymbol{A}' = \boldsymbol{A} + \nabla\lambda$ we obtain

$$\text{curl curl}\ \boldsymbol{A}' = k\ \text{curl}\ \boldsymbol{A}'$$

which requires that

$$\text{curl}\ \boldsymbol{B}' = k\boldsymbol{B}'$$

i.e. the magnetic field $\boldsymbol{B}$ satisfies the same force-free condition (2) under gauge transformations.

Secondly, we remark that Salingaros has taken equation (2) to be a general field equation whose solutions should be covariant with respect to general transformations. This is too stringent and may not correspond to any real field. We note that the general field equations to be satisfied by the magnetic field are Maxwell's equations. It is well known that Maxwell's equations satisfy the property of general covariance necessary to represent physical fields. In fact, the questions raised by Salingaros regarding parity and covariance of solutions in relation to Maxwell's equations and the force-free condition have already been discussed by Chu (1983) in reply to Lee (1983). In studying physical models involving magnetic fields which are locally force-free in some region, we should confine our attention to solutions of Maxwell's equations.

Indeed, it is known that a magnetic field cannot be force-free everywhere in space (Ferraro and Plumpton 1966). A non-vanishing magnetic field, which must satisfy Maxwell's equations everywhere, can satisfy the force-free condition only in a restricted domain. Usually this is sufficient for physical applications, such as models of stellar atmospheres and spiral arms of galaxies. Thus the solutions displayed in Ferraro and Plumpton (1966) would be relevant for physical models.

References

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